Spectra of Length and Area in $(2+1)$ Lorentzian Loop Quantum Gravity

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Abstract

We study the spectrum of the length and area operators in Lorentzian loop quantum gravity, in 2+1 spacetime dimensions. We find that the spectrum of *spacelike* intervals is *continuous*, whereas the spectrum of *timelike* intervals is *discrete*. This result contradicts the expectation that spacelike intervals are always discrete. On the other hand, it is consistent with the results of the spin foam quantization of the same theory.

I. INTRODUCTION

A characteristic feature of the loop approach to quantum gravity is the discrete spectrum of several geometrical quantities. In four spacetime dimensions (4d), the easiest geometrical operator to diagonalize is the area, and its eigenvalues turn out to be discrete [1, 2]. This is generally expected to be true in the Euclidean as well as in the Lorentzian case, since the two theories can be formulated using the same kinematics, differing only in the Hamiltonian constraint. In three spacetime dimensions (3d), the length operator plays a role analogous to the area operator in 4d [3]. For the 3d Euclidean case, the eigenvalues of the length are discrete [3]. In this letter we point out that, surprisingly, the spectrum of the length appears to become continuous in the Lorentzian case.

This difference can be traced to the fact that in the usual tetrad/triad formulation the canonical structure of general relativity is quite different in 4d or in 3d. In 4d the Lagrangian internal gauge group is $SO(4)$ in the Euclidean case and the Lorentz group $SO(3,1)$ in the Lorentzian case. However, both these groups are reduced to a $SO(3)$ subgroup when going to the canonical theory, in order to solve the second class constraints. (An alternative approach where the internal Lorentz group is not broken has been recently explored in [4, 5, 6]). In loop quantum gravity [7], the area operator turns out to be given by the Casimir of the internal gauge algebra. The Casimir of the $so(3)$ algebra has discrete eigenvalues, yielding a discrete spectrum for the area. In 3d, on the contrary, the internal gauge group of the canonical theory is the same as in the Lagrangian theory: it is $SO(3)$ in the Euclidean case and $SO(2,1)$ in the Lorentzian case. In 3d it is the length operator that turns out to be given by the Casimir of the internal gauge algebra. (In this sense length in 3d is analogous to area in 4d.) In the Euclidean case, the Casimir of $so(3)$ has discrete eigenvalues, yielding discrete length. But in the Lorentzian case, the Casimir of the $so(2,1)$ algebra has discrete as well as continuous eigenvalues. The Casimir has opposite sign in the two cases, and a careful tracking of the sign leads to the surprising result that spacelike intervals are not quantized, while timelike intervals are. This is contrary to the expectation that discreteness is a feature of geometrical quantities at fixed time.

Intuitively, one can visualize the geometry of the situation as follows. In 3d there is one timelike direction and two spacelike directions. In the Lorentz algebra there is –correspondingly– one rotation and two boosts. The timelike direction is naturally associated with the rotation. In turn, the rotation (as opposite to the boosts) is associated with the discrete spectrum. The timelike/spacelike character of the $so(2,1)$ unitary representations was also emphasized by Witten in [8].

An analogous exchange (spacelike $\leftrightarrow$ continuous and timelike $\leftrightarrow$ discrete instead of the contrary) was observed by Barrett and Crane in [9] in a covariant spin foam treatment of 4d quantum gravity, as well as in [10] in a similar context. In 3d, the same kind phenomenon was observed by 't Hooft in the context of the canonical quantization of a

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tesselated (Cauchy) surface [11] (although it is also argued that some imaginary part of space-like distances should be discrete) and then again in the context of covariant spin foam models [12]. One might then have suspected that this exchange is a feature of the spin foam approach, in contrast with loop canonical results. The result in this paper rules out this idea, and shows that in 3d there is a remarkable consistency between the results of the spin foam approach (in [12, 13]) and of the loop approach (here). This is similar to results obtained through a covariant treatment of the canonical theory in 4d [5, 6, 14], where a continuous spectrum for spacelike intervals was derived in contrast with the usual result obtained in Loop Quantum Gravity. On the other hand, it is in contrast with the compact group approach to 2+1 loop gravity, as developped in [15], which uses a Wick rotation and derives a discrete spectrum for space-like distances.

II. $2+1$ LOOP QUANTUM GRAVITY

A. Canonical structure

In 3d, general relativity can be formulated as follows. The gravitational field is represented by an $SO(2,1)$ connection $A^i_\mu(x)$ and a triad $e^i_\mu(x)$. Here $\mu = 0,1,2$ is a space-time (co-)tangent index, and $i = 0,1,2$ is an internal index, labelling a basis in the $so(2,1)$ algebra. We will be working in a space-time of signature $(-++)$, so that we raise and lower internal indices using the flat metric $\eta_{ij} = \mathrm{diag}[-++]$. The action is then given in terms of the triad and the field strength $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + \eta^{ij}\epsilon_{jkl}A^k_\mu A^l_\nu$ (ϵ_{ijk} is the completely antisymmetric object) by

$$S(A,e) = \frac{1}{G}\int \mathrm{tr}(e\wedge F) = \frac{1}{G}\int d^3x\ \eta_{ij}\ \epsilon^{abc}\ e^i_a\ F^j_{bc}, \tag{1}$$

where G is a (rescaled) Newton constant. We can perform the usual Hamiltonian analysis, by choosing x^0 as the time evolution parameter and $x^a = (x^1, x^2)$, as coordinates of the initial surface Σ, which we take closed and orientable. Then the action can be decomposed as

$$\begin{aligned} S &= \frac{1}{G}\int dt \int_\Sigma dx^a\ \left(\eta_{ij}\,\epsilon^{ab}\ e^i_a\left(\partial_0 A^j_b - \partial_b A^j_0 + \eta^{jk}\epsilon_{klm}A^l_b A^m_0\right) + \eta_{ij}\epsilon^{ab}e^i_0\ F^j_{ab}\right) \\ &= \frac{1}{G}\int dt \int_\Sigma dx^a\ \left(\epsilon^{ab}\eta_{ij}e^i_a\partial_0 A^j_b + A^i_0\epsilon^{ab}(\eta_{ij}\partial_b e^j_a + \epsilon_{ijk}e^j_a A^k_b) + e^i_0\eta_{ij}\epsilon^{ab}F^j_{ab}\right), \end{aligned} \tag{2}$$

where $\epsilon^{ab} = \epsilon^{0ab}$. From this expression we can read out that the canonical variables are $A^i_a(x)$, and their conjugate momenta are $\pi^a_i(x) = \frac{1}{G}\eta_{ij}\epsilon^{ab}e^j_b(x)$. The fundamental Poisson bracket is therefore

$$\{A^i_a(x), e^j_b(y)\} = G\ \epsilon_{ab}\,\eta^{ij}\,\delta^{(2)}(x,y). \tag{3}$$

The Lagrange multipliers A^i_0 and e^i_0 enforce the constraints $\epsilon^{ab}\mathcal{D}_a e^i_b = 0$ and $F^i_{ab} = 0$, respectively. The first one –the *Gauss law*– generates the $SO(2,1)$ gauge transformations. The second one forces the curvature to be flat. It generates a variation of the frame field:

$$\left|\begin{array}{lcl} \delta e^i_a &=& \mathcal{D}_a\lambda^i \\ \delta A^i_a &=& 0. \end{array}\right. \tag{4}$$

When the frame field is non-degenerate, the second constraint can be decomposed in a vector constraint imposing invariance under 2d space diffeomorphism and a scalar constraint (or Hamiltonian constraint) [16]. More precisely, let us introduce the (co-)frame field conjugate to the connection:

$$E^a_i = \epsilon^{ab}\eta_{ij}e^j_b, \tag{5}$$

and the normal density vector

$$E^i = \frac{1}{2}\epsilon^{ijk}\epsilon_{ab}E^a_j E^b_k. \tag{6}$$

Using the facts that $\eta_{ij}E^iE^j = -\det(^2g)$ is the determinant of the 2-metric (the minus sign is due to the Lorentzian signature of the metric η) and that $E^a_i E^i = 0$, we can decompose the flatness constraint as follows [16]

$$N^iF^i = N^aV_a + NH. \tag{7}$$

N^a and N are respectively the Shift and the Lapse, defined as:

$$N^i = N^a \epsilon_{ab} E^b_i + N \frac{E^i}{\sqrt{\det(^2g)}} \qquad \Leftrightarrow \qquad N^a = \epsilon_{ijk} \frac{E^i E^a_j}{\det(^2g)} N^k \text{ and } N = \frac{N^i E^i}{\sqrt{\det(^2g)}}. \tag{8}$$

V_a is a vector constraint enforcing the invariance under space diffeomorphisms and H is the Hamiltonian constraint:

$$\begin{cases} V_a &= \quad F^i_{ab} E^b_i \\ H &= \frac{1}{2} \epsilon_{ijk} F^i_{ab} \frac{E^a_j E^b_k}{\sqrt{\det(^2g)}} \end{cases} \tag{9}$$

In the present paper, we study the kinematical structure of the quantum theory, which describes the quantum geometry. We shall not deal with the dynamics.

B. Loop Quantization

A quantum theory is defined by a space of quantum states and an algebra of operators. In loop quantum gravity, the (kinematical) states of the geometry are chosen to be cylindrical functions of the connection A. A cylindrical function is determined by an oriented graph Γ (with E edges) and a function ψ on $(SO(2,1))^E$. It is defined as

$$\Psi_{\Gamma,\psi}(A) = \psi(U_1(A), \ldots, U_E(A)) \tag{10}$$

where

$$U_e(A) = \mathcal{P} \exp \left(\int_e ds \, \dot{\alpha}^a(s) \, A^i_a(\alpha(s)) \; \tau_i \right). \tag{11}$$

is the holonomy of the connection A along the edge e of the graph. Here τ_i are the three generators of the $so(2,1)$ algebra.

Two basic operators are defined on the space of these functionals. The first is the holonomy of the connection A along any loop. It acts multiplicatively on the functionals of A. The second is the operator value distribution corresponding the field $e^i_a(x)$. This is given by the differential operator

$$e^i_a(x) = -i\hbar G \epsilon_{ab} \eta^{ij} \frac{\delta}{\delta A^j_b(x)}, \tag{12}$$

where $\hbar G$ is the Planck length l_P in three dimensions. The quantum algebra of these operators provides a quantization of their classical Poisson algebra.

Recall the identity [19]

$$\frac{\delta}{\delta A^j_b(x)} U_e(A) = \int_e ds \, \frac{de^b(s)}{ds} \delta^{(2)}(e(s), x) \; U_{e_1(s)}(A)[X^i U_{e_2(s)}](A), \tag{13}$$

where $e_1(s)$ and $e_2(s)$ are the two parts in which e is split by the point x, and X^i is the generator of the left action of $SO(2,1)$ on the functions on the group. Using this, we have immediately the action of the triad field operator density on the cylindrical states. If x is not on Γ, this action vanishes. Assuming for simplicity that x is in the interior of the edge e, it is

$$e^i_a(x) \Psi_{\Gamma,\psi} = -i\hbar G \epsilon_{ab} \eta^{ij} \int_e ds \; \frac{de^b(s)}{ds} \delta^{(2)}(e(s), x) \; \Psi_{\Gamma_s, X^i \psi_s} \tag{14}$$

where Γ_s is the graph Γ with e split into $e_1(s)$ and $e_2(s)$,

$$\psi_s(U_1, ..., U_{e_1(s)}, U_{e_2(s)}, ..., U_E) = \psi(U_1, ..., U_{e_1(s)} U_{e_2(s)}, ..., U_E) \tag{15}$$

and X^i acts on the variable $U_{e_2(s)}$ by left multiplication.

The Gauss constraint imposes the states to be invariant under $SO(2,1)$ gauge transformation of the connection. This implies that the functions ψ must be invariant at the nodes in the following sense

$$\psi(U_1, \ldots, U_E) = \psi(k_{s(1)} U_1 k^{-1}_{t(1)}, \ldots, k_{s(E)} U_E k^{-1}_{t(E)}), \qquad \forall k_v \in SU(1,1), \tag{16}$$

where $s(e)$ is the source node of the edge e and $t(e)$ its target node.

The scalar product on the space of these states is determined by the requirement that real classical quantities be represented by hermitian operators. Observe first that any cylindrical functional $\Psi_{\Gamma\psi}$ determined by a graph Γ can be rewritten as a cylindrical functional determined by a graph Γ' that contains Γ (such that Γ is a subgraph of Γ')

$$\Psi_{\Gamma\psi}(A) = \Psi_{\Gamma'\psi'}(A); \tag{17}$$

indeed, it is sufficient to take ψ' as independent from the edges of Γ' that are not in Γ. Using this, it is clear that we can always write any two cylindrical functionals in terms of the same graph. Using this fact, in the cases in which the gauge group is compact the scalar product between two states is defined by

$$(\Psi_{\Gamma\psi}, \Psi_{\Gamma\psi'}) = \int dU_1...dU_E \ \ \overline{\psi(U_1...U_E)}\ \psi'(U_1...U_E). \tag{18}$$

The holonomy operator acts by multiplication and the reality condition for the connection $\widehat{A_a^i}^\dagger = \widehat{A_a^i}$ is trivially implemented. The hermicity of the frame field operator, implies simply that the operator iX^i in (14) be hermitian, namely that the measure dU_e is invariant under the action of the group. That is, it must be the Haar measure. Notice that if the states are independent from a link e of the graph, the integration in dU_e becomes irrelevant thanks to the normalization of the Haar measure of a compact group: $\int dU_e = 1$.

In our case, however, $SO(2,1)$ is non compact. This implies that more care is required in the definition of the scalar product. A gauge invariant state, in particular, is constant along an orbit of the group, and the integral in (18) diverges along this orbit. Similarly, a moment of reflection shows that the triad operator may send a finite norm state into an infinite norm state. These divergences can be taken care of by restricting the integration in (18) to a suitably chosen subset of integration variables, such that the spurious integrations along gauge orbits are eliminated. The construction amounts to divide out the volume of the gauge group. In [17] it was shown that this can be done systematically.

More precisely, we call a *connection* on a graph Γ the assignment of group elements U_e to each link of the graph and we define $G_\Gamma = G^{\otimes E}/G^{\otimes V}$ (where Γ has E edges and V vertices) the space of the equivalence classes of these connections under the gauge transformation (16). Notice that a function ψ satisfying (16) is a function on G_Γ. It is shown in [17] that the Haar measure naturally defines a measure $d\mu_\Gamma$ on G_Γ. Each graph has then an associated Hilbert space $L^2(G_\Gamma, d\mu_\Gamma)$, and we can replace (18) by the scalar product in this Hilbert space. Holonomy functionals of the connection with support on Γ act by multiplication on this space and this implements the reality condition for the connection. It is then proven in [17] that any gauge invariant operator constructed with the triad field is well defined $L^2(G_\Gamma, d\mu_\Gamma)$ and is hermitian in this measure, which implements the hermicity condition for the frame fields. We refer the reader to [17] for all details[1].

Using the Plancherel decomposition of L^2 functions on the group (with respect to the Haar measure), an orthonormal basis of states in L^2_Γ can be constructed as spin networks and involves the infinite dimensional unitary representations of $SO(2,1)$. Recall that L^2 functions over the group can be expanded over an orthonormal basis provided by irreducible representations of the group. This is called the Plancherel decomposition of the L^2 functions on the group. The representations appearing in this expansion are the ones of the *principal* and *discrete* series of *unitary* representations, and will be described below in Section III.

Then, we can construct an orthonormal basis of gauge invariant states as follows. Once the graph Γ is fixed, we choose a principal unitary $SO(2,1)$ irreducible representation $\mathcal{I}_e$ (entering the Plancherel decomposition) for each edge e of the graph. Contract the representation matrices of the holonomies $U_e^{\mathcal{I}_e}$ in these representations using a $SO(2,1)$ intertwiner at each node (intertwining the representations associated to the edges incident to the node). The resulting function is an $SO(2,1)$ spin network functional. It depends on the graph, the representations associated to the edges and the intertwiners associated to the nodes. The set of these functionals (for all graphs, all choices of irreducible representations and intertwiners out of a basis in the linear space of the intertwiners) form a complete orthonormal (generalized) basis of gauge invariant cylindrical functionals.

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[1] The tricky point (and the difference with the compact group case) in the construction is that the space L^2_Γ for a graph Γ can not be embedded in the space $L^2_{\Gamma'}$ associated to a bigger graph. As a consequence, the full space of quantum states of geometry, obtained by gluing these L^2_Γ spaces together (summing over graphs), can not be obtained as a projective limit anymore (as in the compact group case) and, therefore, doesn't seem to be a L^2 space. On the other hand, it is still a Hilbert space, with a structure similar to a Fock space.

C. Length operator

The length of a differential curve $c : \tau \in [0,1] \to c(\tau) \in \Sigma$ is given by

$$L_c = \int_{[0,1]} d\tau \, \sqrt{\dot{c}^a(\tau)\,\dot{c}^b(\tau)\; g_{ab}(c(\tau))} = \int_{[0,1]} d\tau \, \sqrt{\dot{c}^a(\tau)\,\dot{c}^b(\tau)\; \eta_{ij}\, e^i_a(c(\tau))\, e^j_b(c(\tau))}. \tag{19}$$

For simplicity of the notations, let us introduce the vector

$$e^i(c(\tau)) = e^i_a(c(\tau))\dot{c}^a(\tau). \tag{20}$$

In the study of the length, we will restrict ourselves to the case in which the norm $\eta_{ij}e^ie^j$ of the vector e^i doesn't change sign along the curve. That is, we require the curve to be entirely time-like or entirely space-like. Notice that in the case of a time-like curve, that is $\eta_{ij}e^ie^j < 0$, there is another gauge invariant quantity besides the norm of e^i: the sign of e^0. This sign is invariant under $SO(2,1)$ and registers the time orientation, past or future, of the curve.

The length of a space-like curve ($\eta_{ij}e^ie^j > 0$) is be defined by (19) as a real number. In the case of a time-like curve ($\eta_{ij}e^ie^j < 0$) it is convenient to define an oriented real time interval taking into account the time orientation as

$$T_c = \mathrm{sign}(e^0) \int_{[0,1]} dt \, \sqrt{|\eta_{ij}e^ie^j|}. \tag{21}$$

The quantum operator representing the classical length is obtained replacing the triad field $e^i_a(x)$ with the corresponding quantum operator (12) in these expressions. We now study the action of this length operator on spin network states, following the example of the area operator in $3+1$ loop quantum gravity [2, 18]. Our concern here is not in the details of the regularization of this operator, which have been extensively discussed elsewhere, but just on the particular features of the Lorentzian 2+1 case.

Consider a curve c and a spin network state such that the curve and the underlying graph intersect only once and not at a node of the graph. We consider the action of the length operator of the curve c on this state. (What follows can be easily generalized to multiple intersections and to intersections at nodes.) Call γ the edge of the spin network intersected by the curve c. Let $\mathcal{I}$ be the irreducible representation associated to γ.

The action of $e^i(x)$ on the spin network state inserts the generator X^i in the state. The action of X^i on the representation $\mathcal{I}$ is given by the generator $X^i_{\mathcal{I}}$ in this representation. We obtain easily

$$L_c\Psi^{(\mathcal{I})} = \hbar G \left[\int_c d\tau \sqrt{\left(\int_\gamma ds \, \epsilon_{ab}\dot{c}^a(\tau)\frac{d\gamma^b}{ds}\delta^{(2)}(\gamma(s), c(\tau)) \right)^2 \left(-\eta_{jk} X^j_{(\mathcal{I})} X^k_{(\mathcal{I})} \right)} \;\right] \;\; \Psi^{(\mathcal{I})}, \tag{22}$$

which shows that the spin networks are eigenvectors of the length operator. The integral quantity

$$\int_c d\tau \int_\gamma ds \left| \epsilon_{ab}\dot{c}^a(\tau)\frac{d\gamma^b}{ds}\delta^{(2)}(\gamma(s), c(\tau)) \right| \tag{23}$$

is the number of intersections between the curve c and the edge γ of the spin network graph. More precisely, $\epsilon_{ab}\dot{c}^a(\tau)\gamma'^b(s)$ is the Jacobian of the transformation between orthonormal coordinates (x_1, x_2) and the local coordinates (τ, s). This works only when the curve c and the edge γ are not tangential else the action of L_c vanishes. Here we have assumed the intersection number to be 1, so that the action of the length operator reduces to

$$L_c \; \Psi^{(\mathcal{I})} = \hbar G \, \sqrt{-\eta_{jk} X^j_{(\mathcal{I})} X^k_{(\mathcal{I})}} \;\; \Psi^{(\mathcal{I})} = \hbar G \, \sqrt{q^{(\mathcal{I})}} \; \Psi^{(\mathcal{I})}. \tag{24}$$

where $Q = -\eta_{ij}X^iX^j$ is the Casimir operator for $SO(2,1)$ and $q^{(\mathcal{I})}$ is its value in the representation $\mathcal{I}$. This gives the length spectrum of $2+1$ gravity in its loop quantized version, which we study in the next section. Depending on the sign of $q^{(\mathcal{I})}$, we get either a space-like length, a time-like length or a null curve. If the case of a time-like or null curve, we further need to specify the orientation observable sign(e^0). This should correspond to some data encoded in the representation $\mathcal{I}$. More precisely, it should be the sign of the operator $\widehat{e^0}$, which should thus have a spectrum with only positive eigenvalues or only negative eigenvalues.

The length operator of a curve acting on a spin network which it intersects many times is given by a contribution for each edge intersected.

$$L_c \; \Psi = l_P \sum_{e|e\cap c \neq \emptyset} \sqrt{q^{(\mathcal{I}_e)}} \, \Psi. \tag{25}$$

III. REPRESENTATIONS OF $SO(2,1)$ AND THE LENGTH SPECTRUM

We are interested in the group $SO(2,1)$, the Lorentz symmetry group of $2+1$ gravity. Its Lie algebra is of dimension 3 and generated in its fundamental representation by the following three matrices [2]:

$$\tau_0 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \frac{i}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{26}$$

The commutation relations between these generators are:

$$[\tau_0, \tau_1] = -\tau_2 \qquad [\tau_1, \tau_2] = \tau_0 \qquad [\tau_2, \tau_0] = -\tau_1. \tag{27}$$

One can check that these are the right signs for the symmetry group $SO(2,1)$ of a $(-,+,+)$ Lorentz space. Let X_i be the generators of a linear representation of the group. They are linear operators that satisfy

$$[X_0, X_1] = -X_2 \qquad [X_1, X_2] = X_0 \qquad [X_2, X_0] = -X_1. \tag{28}$$

It is important not to confuse the hermicity properties of the matrices τ_i and the hermicity properties of the X_i. As we have discussed above, the representations playing a role in quantum gravity are the ones appearing in the Plancherel decompositions of the L^2 functions with respect to the Haar measure. These representations are unitary i.e the linear operators iX_i are hermitian. Indeed, these are (up to constants) precisely the quantities corresponding the triad field operator, and their hermicity reflects the fact that the triad field is real.

It is useful to study the algebra using the operators H and $J_\pm$ defined as:

$$H = -iX_0 \qquad J_\pm = \pm X_1 + iX_2 \tag{29}$$

with the commutation relations:

$$[H, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = -2H. \tag{30}$$

The difference with the real algebra $so(3)$ is the minus sign in the second commutation relation. The Casimir operator is

$$Q = (X_0)^2 - (X_1)^2 - (X_2)^2 = -H^2 + \frac{1}{2}(J_+J_- + J_-J_+). \tag{31}$$

The reality conditions expressing that the iX_i are hermitian are expressed as:

$$H^\dagger = H \qquad (J_\pm)^\dagger = J_\mp. \tag{32}$$

A. Representations of $SO(2,1)$

The representations of $SO(2,1)$ can be studied in the same type of basis as for $SO(3)$. Indeed it is easy to check that

$$\left\{ \begin{array}{rcl} H|m\rangle & = & m|m\rangle, \\ J_+|m\rangle & = & (q+m(m+1))^{1/2}|m+1\rangle, \\ J_-|m\rangle & = & (q+m(m-1))^{1/2}|m-1\rangle \end{array} \right. \tag{33}$$

gives a representation of $SO(2,1)$ on the space spanned by the vectors $\{|m\rangle\ ,\ m \in \mathbf{Z}\}$. The the parameter q gives the value of the Casimir operator. [3]

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[2] In fact, this is the fundamental representation for $SU(1,1)$, which is the double cover of $SO(2,1)$. In the present paper, we will use only the group $SO(2,1)$ for its representation theory is simpler. However, all the results can be obviously extended to the case of $SU(1,1)$ and we present its representation theory in appendix A. Let us nevertheless point out that $SU(1,1)$ is not the universal cover of $SO(2,1)$ unlike the Euclidean case where $SU(2)$ was actually the universal cover of $SO(3)$.

[3] If we replace m by $m+1/2$ everywhere in (33) we get a representation of $SU(1,1)$. If we replace m by $m+\alpha$, $0<\alpha<1$ we get a representation of the universal cover of $SO(2,1)$

The unitary representations are infinite dimensional since $SO(2,1)$ is non-compact. Their Casimir operator is hermitian $Q^\dagger = Q$. This implies that q is real. Let us consider the different representations obtained for real values of the parameter q.

Consider first the case of a negative Casimir $q \leq 0$. For generic values, the representation obtained is irreducible. However $(q + m(m+1))$ can take some negative values, and this contradicts the unitarity relation $(J_\pm)^\dagger = J_\mp$.

For the special values $q = -n(n-1) \leq 0$, with $n \in \mathbb{N}^*$, $(q + m(m+1))$ vanishes for values $m = n-1$ and $m = -n$. Therefore, the representation is not irreducible. In facts, it decomposes into 3 representations.

There are "intermediate" representations, called V^n, which are finite dimensional. They are spanned by the vectors $\{|m\rangle\ ,\ -(n-1) \leq m \leq (n-1)\}$. They are the same representation as the finite irreducible (spin $j = n-1$) representation of $SO(3)$. However $(q+m(m+1)) < 0$ and we have $(J_\pm)^\dagger = -J_\mp$, which violates the reality conditions: they are not unitary.

The other two representation are infinite dimensional. The upper one $\mathcal{D}_n^+$ is a lowest weight representation spanned by values $m \in n + \mathbb{N}$. The lower one $\mathcal{D}_n^-$ is a highest weight representation spanned by $m \in -(n + \mathbb{N})$. These representations are unitary.

For a positive value of the Casimir $q > 0$, $(q + m(m+1)) = q - 1/4 + (m+1/2)^2$ stays always positive and we get infinite dimensional unitary representations spanned by all $m \in \mathbf{Z}$. $0 < q < 1/4$ labels the exceptional series whereas $q > 1/4$ labels the principal series. The representations of the principal series are denoted $\mathcal{C}_s$, with $q = s^2 + 1/4$.

The unitary irreducible representations $\mathcal{D}_n^+$, $\mathcal{D}_n^-$ and $\mathcal{C}_s$ are the ones coming into the *Plancherel decomposition* of a L^2 function f on the group $SO(2,1)$:

$$\begin{aligned} f(g) \;=\; & \sum_{n\geq 1} (2n-1)\ \mathrm{Tr}(f_n^+ R_{\mathcal{D}_n^+}(g)) + \sum_{n\geq 1} (2n-1)\ \mathrm{Tr}(f_n^- R_{\mathcal{D}_n^-}(g)) \\ & + \int_{s>0} ds\ \frac{\coth(\pi s)}{4\pi s}\ \mathrm{Tr}(f_s R_{\mathcal{C}_s}(g)), \end{aligned} \tag{34}$$

Notice that the continuous representations start at $q = 1/4$ instead of $q = 0$.

B. Length spectrum

The eigenvalues of the length operator associated to a curve are given by the square root of the values of the Casimir operator of the representation carried by the edge that the curve intersects. A continuous representation $\mathcal{C}_s$ has *positive* Casimir and correspond to a space-like length with eigenvalue

$$L_s = \sqrt{s^2 + 1/4}. \tag{35}$$

Notices the gap 1/2. It implies that there exists a minimal space-like length even if the spectrum is *continuous*.

A discrete representations $\mathcal{D}_n^\epsilon$ ($\epsilon = \pm$ and $n \in \mathbb{N}$) has *negative* Casimir and corresponds to a time-like curve. Its past or future orientation $\mathrm{sgn}(e^0)$ is given by ϵ. Indeed, ϵ is the sign of the (eigenvalues of the) generator H, which is the operator quantizing e^0. Then the length spectrum corresponding to a time-like curve will be *discrete*. The eigenvalues of the observable (21) are

$$T_{\epsilon,n} = \epsilon\sqrt{n(n-1)}. \tag{36}$$

Notice that the eigenvalues do not have equal spacing. Notice also that the first discrete representations $\mathcal{D}_{n=1}^\pm$ have vanishing Casimir and length eigenvalue, and thus correspond to a null curve, the sign $\pm$ still corresponding to the past or future orientation of the curve. See Figure 1.

More precisely, the eigenvalues of the length operators are given by any sum of these eigenvalues. Each term of the sum corresponds to one intersection between the graph of the state and the curve c. Note that the gap in the real axis between 0 and 1/2 correspond to a class of unitary representations. However, these representations are not L^2 and have a vanishing Plancherel measure (they do not come in the Plancherel decomposition): they are called the *complementary series* of representations.

C. Variants

Alekseev and al. in the context of loop quantum gravity [20] (for the groups $SO(3)$ and $SU(2)$) and Freidel-Krasnov in the context of spin foams [21] have given some arguments for a possible correction to the above length spectrum.

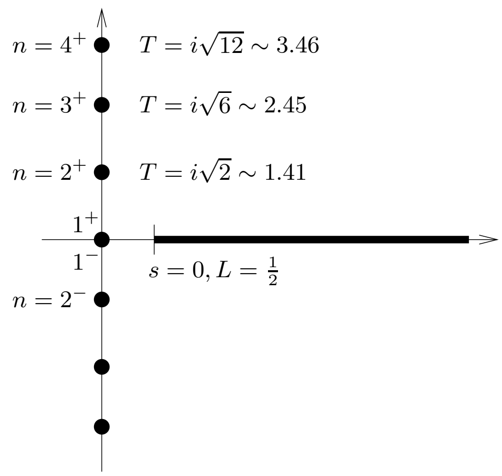

FIG. 1: The spectrum of the length operator.

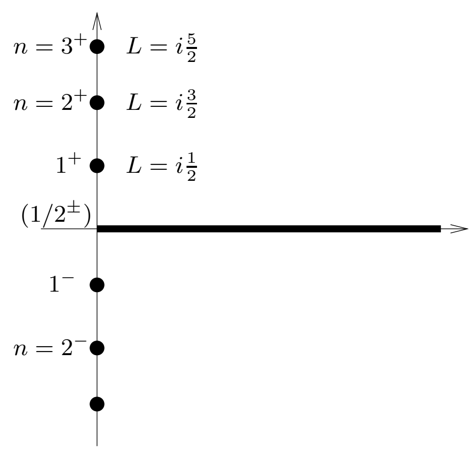

FIG. 2: The spectrum of the symmetric length operator.

We can use the symmetric quantization map for quantizing $\widehat{e^i}$ as the derivation on the Lie algebra (as a vector space) instead of the derivation on the Lie group.[4] Then the Casimir operator q gets shifted to $q - 1/4$ [5] and the length spectrum becomes

$$\left| \begin{array}{rcccc} L_s & = & s & \text{for space-like} & \mathcal{C}_{s>0} \\ T_{\epsilon,n} & = & \epsilon\left(n - \frac{1}{2}\right) & \text{for time-like} & \mathcal{D}^{\epsilon=\pm}_{n\geq 1} \end{array} \right. \tag{37}$$

There is now a minimal time-like interval and no null representation given by the discrete series. Also, the time-like length spectrum becomes equally spaced. On the other hand, the initial gap for the space-like lengths disappear and

[4] As an example, consider the following analogy with the model of the free particle over the group manifold [17]. The theory is defined by the action

$$S = \frac{1}{2}\int dt\, \mathrm{Tr}((g^{-1}\partial_t g)^2).$$

As conjugate variable for the configuration variable g, we can choose either the canonical momentum $p = \partial_t g^{-1}$, which is commutative, or the non-commutative Noether charge $\Pi = g\partial_t g^{-1}$. The last generates the (right) group multiplication and satisfies $\{\Pi_X, \Pi_Y\} = \Pi_{[X,Y]}$, where $\Pi_X = \mathrm{Tr}(X\Pi)$ is the X component of Π.

[5] See [17] for details on $SO(2,1)$ and $SU(1,1)$ and explicit expressions for the Laplacian and the characters.

there is the possibility of a null curve in the limit $s \to 0$. This second length spectrum fits better with the algebraic data and with the Lorentzian 3d Spin Foam picture [12, 13], see also [22] for the asymptotics of 6-j symbols. See Figure 2. In this version of the length spectrum, null representations are also present in the representation theory of $su(1,1)$. There are two extra discrete representations given by $n = 1/2^{\pm}$ (present only for the group $SU(1,1)$ and not for $SO(2,1)$); these representations are unitary but *not* L^2, they are called the *limit of discrete series*.

IV. AREA SPECTRUM

A. Area operator

The area of a surface $\mathcal{S}$ embedded in the canonical (closed and orientable) surface Σ is given by

$$\mathcal{A}_{\mathcal{S}} = \int_{\mathcal{S}} ds^2 \sqrt{\det(^2 g)} \tag{38}$$

where $g_{ab} = e^i_a e^j_b \eta_{ij}$ is the 2-metric on Σ. We now study the quantum operator corresponding to this quantity.

The determinant of the metric can be written as $\det(^2 g) = -\eta^{ij} E_i E_j$ in term of the normal density vector $E_i(x) = \frac{1}{2}\epsilon_{ijk}\epsilon^{ab} e^j_a(x) e^k_b(x)$ introduced in (6). When acting on a spin network, the frame field operator has a non-vanishing action only if x belongs to the graph. When x is in the middle of an edge, the action of the operator is proportional to the tangent $\dot\gamma_a(s)X^i$ of the edge (see (14)). It follows immediately that $\epsilon^{ab}\dot\gamma_a\dot\gamma_b = 0$, vanishes and therefore the operator corresponding to $E_i(x)$ gives zero. The only points at which $E_i(x)$ will have a non-vanishing action are the nodes of the graph: *the area operator has contributions only from the nodes.*

To compute the action of the area operator of a surface $\mathcal{S}$ on a spin network state, we cut up the surface into small bits, each containing at most one node of the spin network. We can thus restrict ourselves to the study of a (elementary) surface containing only one node n of the spin network on which the area operator acts. For simplicity, we also restrict ourselves to the case where the node is 3-valent. To define the area operator, we need to choose an *orientation* for Σ, even though the final result will be independent of the chosen orientation. This corresponds to choosing a consistent *ordering* of the three edges incident on each node of the graph.

The node n has three incident edges $e = 1, 2, 3$ (following the orientation) with $SO(2,1)$ representation $\mathcal{I}_e$. To begin with, define an auxiliary operator acting at the node n by the insertion of some X operators

$$\widetilde{E_i}\Psi^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3} = -\frac{l_P^2}{2}\epsilon_{ee'}\epsilon_{ijk} X^j_{\mathcal{I}_e} X^k_{\mathcal{I}_{e'}} \Psi^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3}, \tag{39}$$

where e, e' are any two edges meeting at v and $\epsilon_{ee'}$ registers the orientation of the two edges around the node. Using the fact that $\vec X_{\mathcal{I}_1} + \vec X_{\mathcal{I}_2} + \vec X_{\mathcal{I}_3} = 0$, one can get a more symmetric expression for $\widetilde{E_i}$ by summing the above expression over the couples of edges (e, e'), with a symmetry factor $1/3$.

We now look at the action of $\widehat{E_i}(x)$ at the node n, a direct computation using (14) gives .

$$\widehat{E_i}(x)\Psi^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3}_v = \alpha(x,v)\ \widetilde{E_i}\Psi^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3}_v, \tag{40}$$

where the geometrical factor is

$$\alpha(x,v) = \sum_{e,e'} \int dsdt\ \delta^2(x,\gamma_e(s))\ \delta^2(x,\gamma_{e'}(t))\ |\epsilon_{ab}\gamma^a_e(s)\gamma^b_{e'}(t)|. \tag{41}$$

We can regulate this factor and see that it is just proportional to $\delta^2(x,v)$. Therefore the area operator acts on the spin network as:

$$\begin{aligned}
\mathcal{A}_{\mathcal{S}}\,\Psi^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3} &= \sqrt{-\eta^{ii'}\widetilde{E_i}\widetilde{E'_i}}\,\Psi^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3} && (42)\\
&= l_P^2\sqrt{-\frac{1}{4}\eta^{ii'}\epsilon_{ijk}\epsilon_{i'j'k'}X^j_{\mathcal{I}_1}X^k_{\mathcal{I}_2}X^{j'}_{\mathcal{I}_1}X^{k'}_{\mathcal{I}_2}}\ \Psi^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3} \\
&= l_P^2\frac{1}{2}\sqrt{\Big((\vec X_{\mathcal{I}_1})^2(\vec X_{\mathcal{I}_2})^2 - (\vec X_{\mathcal{I}_1}.\vec X_{\mathcal{I}_2})^2\Big)}\Psi^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3} \\
&= l_P^2\frac{1}{2}\sqrt{|\vec X_{\mathcal{I}_1}\wedge\vec X_{\mathcal{I}_2}|^2}\Psi^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3}. && (43)
\end{aligned}$$

Using the fact that $\vec{X}_{\mathcal{I}_1}+\vec{X}_{\mathcal{I}_2}+\vec{X}_{\mathcal{I}_3}=0$, we can express the above factor in terms of the Casimir operators $q^{\mathcal{I}_\alpha}$ of the three representations. Therefore the area operator is diagonal in the spin network basis with eigenvalue:

$$\mathcal{A}_{\mathcal{S}} = l_P^2 \frac{1}{2}\sqrt{q^{\mathcal{I}_1}q^{\mathcal{I}_2}-\frac{1}{4}(q^{\mathcal{I}_3}-q^{\mathcal{I}_1}-q^{\mathcal{I}_2})^2}. \tag{44}$$

This fits with the definition of the area of a geometrical triangle defined by the length of its three edges given by $L_\alpha=\sqrt{q^{\mathcal{I}_\alpha}}$. More precisely, the above formula can be rewritten as

$$\mathcal{A}_{\mathcal{S}} = l_P^2 \frac{1}{4}\sqrt{(L_1+L_2+L_3)(-L_1+L_2+L_3)(L_1-L_2+L_3)(L_1+L_2-L_3)}. \tag{45}$$

This is consistent with the results on the length operator since $L_\alpha=\sqrt{q^{\mathcal{I}_\alpha}}$ is precisely the length spectrum associated to the spin network edge α.

In conclusion, a labelled spin network has a geometrical interpretation as a two-dimensional discrete triangulated manifold. The faces have finite area and are dual to the nodes of the graph. Faces are separated by edges with finite length, dual to the link of the graph.

B. $SO(2,1)$ Intertwiners and Area spectrum

To find the spectrum of the area operator explicitly, we need to characterize the admissible nodes i.e the possible triplets of representations $(\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3)$. This corresponds to the existence of an intertwiner between these three representations. Consider the decomposition of the tensor product of two $SO(2,1)$ representations. Not all kinds of principal representations show up in the decomposition of the tensor product of two principal representations. We have (see [13], where explicit expressions of the corresponding Clebsch-Gordon coefficient are given)

$$\mathcal{D}^{\pm}_{n_1}\otimes\mathcal{D}^{\pm}_{n_2} = \bigoplus_{n\geq n_1+n_2}\mathcal{D}^{\pm}_{n}, \tag{46}$$

$$\mathcal{D}^{+}_{n_1}\otimes\mathcal{D}^{-}_{n_2} = \bigoplus_{n=1}^{n_1-n_2}\mathcal{D}^{+}_{n}\oplus\bigoplus_{n=1}^{n_2-n_1}\mathcal{D}^{-}_{n}\oplus\int ds\ \mathcal{C}_s, \tag{47}$$

$$\mathcal{D}^{\pm}_{n_1}\otimes\mathcal{C}_{s_2} = \bigoplus_{n\geq 1}\mathcal{D}^{\pm}_{n}\oplus\int ds\ \mathcal{C}_s, \tag{48}$$

$$\mathcal{C}_{s_1}\otimes\mathcal{C}_{s_2} = \bigoplus_{n\geq 1}\mathcal{D}^{+}_{n}\oplus\bigoplus_{n\geq 1}\mathcal{D}^{-}_{n}\oplus 2\int ds\ \mathcal{C}_s. \tag{49}$$

Precisely as in the Lorentzian 3d spin foam model [12, 13], these decomposition rules can be interpreted as describing the relations between equivalence classes of triangles (under the action of $SO(2,1)$) in Minkovski 3d space. Equivalently, they can be interpreted as sum rules of 3-vectors. More precisely, we can associate the $\mathcal{D}^{\pm}_n$ representations to future or past oriented time-like vectors with norm $L_n=n-1/2$ and the $\mathcal{C}_s$ to space-like vectors with norm $L_s=s$. Notice that this is precisely the association emerged from the spectrum of the length operator. Then, equation (46) corresponds to the fact that summing two time-like vectors (with the same orientation) gives a time-like vector of the same orientation. Furthermore, the sum rule respects the anti-triangular inequality $L_n\geq L_{n_1}+L_{n_2}$. Similarly, equation (49) corresponds to summing triangles formed with space-like edges. The result can be space-like or time-like and there is no (anti-)triangular inequality. [6].

Finally, the area operator eigenvalues give precisely the area of the different types of triangles obtained by summing two vectors as described by these tensor product decomposition rules.

---

[6] Notice that this implies that restricting the theory to the continuous series of representations in order to deal with solely space-like surfaces does not work, unless one also imposes by hand a triangular inequality between the continuous representations, which does not seem very natural from the point of view of representation theory.

V. CONSIDERATIONS

- Our result is not definitive since we have considered only the kinematics of the theory, and not its dynamics. It is not unconceivable that the dynamics could constraint the representation of the operator algebra in some unexpected way. Furthermore, questions remain open on the definition of the full Hilbert space of the theory for non compact groups [17].

- One may wonder how the length operator can have eigenvalues that correspond to both signatures. Since we use the canonical formalism, the curve c lives on the initial value surface. If this is spacelike, how can the curve be timelike? The answer is the following. In the canonical formalism considered, we have never imposed the condition that the metric be spacelike on the initial surface. In fact, the canonical formalism is rather flexible in this regard. In 4d, one usually breaks down the Lorentz group to a three dimensional rotation group. In doing so, one gauge fixes certain components of the tetrad to fixed values (with a well defined sign), and this forces the remaining components, which form the triad, to be spacelike. Nothing similar happens in the canonical formulation of the 3d theory considered here. Therefore, unless one explicitly imposes so, the initial value surface has no determined signature.

- We recall that the length, as the area in 4d, is not a gauge invariant operator, and its quantization has to be properly interpreted as an indication of the corresponding quantization of a suitable quantity defined intrinsically by the dynamical variables themselves, as physical geometrical quantities measured in the laboratory always are. In general, the simplest way to do so is to couple dynamical matter to the gravitational field and use this matter as a physical reference frame [23, 24]. This also explains how the rich structure given by the length operators can be read out from the relatively simple 3d theory, which is topological, and has only a finite number of physical gauge invariant operators. In other words, what we are really exploring here is the non-gauge-fixed level of the theory, which describes the gravitational field as seen by a physical reference system [25].

- We do not measure lengths directly as numbers: numbers are given by ratios between physical lengths. For instance, by the number of times a rod fits into an interval. One may thus wonder whether the sign or the imaginary character of the interval has any importance by itself. The answer is of course not. The imaginary unit simply keeps track of the distinction between the two kind of intervals, which are fundamentally distinguished from each other by their relations, namely by the different way in which they fit into a Minkowski (or a locally Minkowskian) space. It is interesting to notice that these relations between intervals are in fact reproduced by the $su(1,1)$ representation theory. Spacelike and timelike intervals sum among themselves differently, and this is reflected in the way direct products of representations can be decomposed. This works if we identify timelike intervals with discrete representation (plus or minus, according to future and past) and spacelike ones with continuous representations. This is illustrated in the previous section and the Appendix. For instance, the sum of two future timelike vectors can only be a future timelike vector. Accordingly, the direct sum of two representations in the $\mathcal{D}_n^+$ series contains only representations of the $\mathcal{D}_n^+$ series. This fact reinforces the idea that the discrete representations are naturally timelike and the continuous ones are "naturally" spacelike.

- Using the correspondence between Chern-Simons theory and 3 dimensional gravity, we expect the introduction of a cosmological constant $\Lambda > 0$ to deform the group structure to a quantum group structure $U_q(SU(1,1))$, with the deformation parameter q being related to the cosmological length $L = 1/\sqrt{\Lambda}$. The representations of interest are still discrete or continuous. The novelty is that the continuous representations for $q = e^{-h}$ admit an infrared cutoff given by $\frac{\pi}{2h}$ [26], so that no spacelike length can be bigger than $L = \frac{\pi}{2h} L_P$, which is identified to the cosmological scale or, in other words, the distance to the cosmological horizon. This is consistent with the physical intuition that no information is accessible behind the horizon. This way, we expect a relation of the type $q \sim e^{-\sqrt{\Lambda}}$, or more precisely $q = e^{-\frac{\pi L_P}{2L}}$. Similar considerations have recently been developed in a 4-dimensional spin foam approach [27].

- We have introduced two length operators. The first one corresponds to the usual way of quantizing geometrical operators in loop quantum gravity. The spectrum of this operator has both a minimal timelike distance *and* a minimal spacelike distance. This may be surprising, given that the spacelike spectrum is continuous. Also, representations corresponding to the null directions appear in the spectrum of this operator.[7] Finally, the

[7] It might be interesting to notice that using Hod's [28] correspondence principle between quasi normal modes and quantization of black hole area fluctuations, it has been recently argued [29] that the minimum allowed spin relevant for the black hole horizon might be 1. This might perhaps be related with the fact that the spin 1 representation correspond to null directions.

timelike spectrum is not equally spaced.[8] On the other hand, the second operator issued from the symmetric quantization map, agrees with the spin foam computations and the radius of the coadjoint orbits. In this case the timelike spectrum is discrete (there is a minimal length) and equally spaced (moreover, when using $SU(1,1)$ instead of $SO(2,1)$, we allow spin $n \in \mathbb{N}+1/2$ and the difference between two consecutive length values becomes exactly the minimum allowed length), and the spacelike spectrum is continuous and has no initial gap.

- As we mentioned at the end of the introduction, the result in this letter shows that in 3d there is consistency between spin foam [13] and loop results. The situation is still unclear in 4d, where there is an apparent sign discrepancy between spin foam [9, 10] and loop [1, 2] results. In 4d, so far the focus has mostly been on the absolute value, and not on the sign, of quantum geometrical quantities; a detailed investigation of the signature of the area in the quantum regime, and a careful comparison of the spin foam and canonical results, would be of interest. The analysis of the covariant canonical structure of general relativity recently completed in [6] might be a useful step in this direction.

APPENDIX A: REPRESENTATIONS OF $SU(1,1)$

Here we extend the results on the geometrical operators to the group $SU(1,1)$, the double cover of $SO(2,1)$. Just as when extending $SO(3)$ to $SU(2)$, this extension doubles the number of representations and introduces a parity.

In the principal series of $SU(1,1)$, there are two series of continuous representations $\mathcal{C}_s^\epsilon$ where $\epsilon = 0, 1/2$ is the parity and s a positive real number. The Casimir is $q = s^2 + 1/4 > 0$ and the set of weights m is formed by the integers or the half-integers depending on the parity of the representation. There are two series of discrete representations $\mathcal{D}_n^\pm$ labelled by a half-integer n larger than 1. The Casimir is $q = n(1-n) < 0$ and the set of weights m is $n + \mathbb{N}$ for the positive series, and $-(n+\mathbb{N})$ for the negative one. The Plancherel formula for a function $f \in L^2(SU(1,1))$ reads:

$$
\begin{aligned}
f(g) \;=\; & \int_{s>0} ds\, \frac{\coth(\pi s)}{4\pi s}\, \mathrm{Tr}(f_s^0 R_{\mathcal{C}_s^0}(g)) + \int_{s>0} ds\, \frac{\tanh(\pi s)}{4\pi s}\, \mathrm{Tr}(f_s^0 R_{\mathcal{C}_s^{1/2}}(g)) \\
& + \sum_{n\geq 1} (2n-1)\, \mathrm{Tr}(f_n^+ R_{\mathcal{D}_n^+}(g)) + \sum_{n\geq 1} (2n-1)\, \mathrm{Tr}(f_n^- R_{\mathcal{D}_n^-}(g)).
\end{aligned}
\tag{A1}
$$

The Casimir (shifted by one fourth) still give the (square of) length associated to the edge labelled by the representation. The representations $n = \frac{1}{2}^\pm$ are unitary but do not enter the Plancherel decomposition. Physically, their corresponding length is 0 and they correspond to null edges.

As for the area operator and the admissible nodes, there is not much that changes. The only difference is that one must take care of the parity. The tensor product decomposition reads:

$$
\mathcal{D}_{n_1}^\pm \otimes \mathcal{C}_{s_2}^{\epsilon_2} = \bigoplus_{n\geq n_{min}} \mathcal{D}_n^\pm \oplus \int ds\, \mathcal{C}_s^\epsilon, \tag{A2}
$$

$$
\mathcal{C}_{s_1}^{\epsilon_1} \otimes \mathcal{C}_{s_2}^{\epsilon_2} = \bigoplus_{n\geq n_{min}} \mathcal{D}_n^+ \oplus \bigoplus_{n\geq n_{min}} \mathcal{D}_n^- \oplus 2\int ds\, \mathcal{C}_s^\epsilon \tag{A3}
$$

$$
\mathcal{D}_{n_1}^\pm \otimes \mathcal{D}_{n_2}^\pm = \bigoplus_{n\geq n_1+n_2} \mathcal{D}_n^\pm, \tag{A4}
$$

$$
\mathcal{D}_{n_1}^+ \otimes \mathcal{D}_{n_2}^- = \bigoplus_{n=n_{min}}^{n_1-n_2} \mathcal{D}_n^+ \oplus \bigoplus_{n=n_{min}}^{n_2-n_1} \mathcal{D}_n^- \oplus \int ds\, \mathcal{C}_s^\epsilon. \tag{A5}
$$

[8] On the connection between the fact that the spectrum of the area is not equally spaced and the Hawking thermal radiation see [30].

where $n_{min} = 1$ and $\epsilon = 0$ if $n_1 + n_2$ is an integer, $n_{min} = 3/2$ and $\epsilon = 1/2$ otherwise.

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